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SCALAR CURVATURE FUNCTIONS IN A CONFORMAL CLASS OF METRICS AND CONFORMAL TRANSFORMATIONS

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ABSTRACT. This article addresses the problem of prescribing the scalar curvature in a conformal class. (For the standard conformal class on the 2-sphere, this is usually referred to as the Nirenberg problem.) Thanks to the action of the conformal group, integrability conditions due to J. L. Kazdan and F. W. Warner are extended, and shown to be universal. A counterexample to a conjecture by J. L. Kazdan on the role of first spherical harmonics in these integrability conditions on the standard sphere is given. Using the action of the conformal groups, some existence results are also given.

0. Statements of results. Recently, the study of functions which, on a connected C^∞ n-manifold M without boundary, are scalar curvatures of complete Riemannian metrics has drawn special attention among both analysts and geometers. Decisive steps were taken by J. L. Kazdan and F. Warner (cf. [**7**] for an update with bibliography), R. Schoen and S. T. Yau (cf. [**16**]), and M. Gromov and H. B. Lawson (cf. [**6**]). Nevertheless, one among the oldest questions related to scalar curvature functions remains unsolved, the Nirenberg problem, namely "describe all curvature functions on the 2-sphere conformal to a standard one." In this article, we shall address this problem. Since it concerns the most familiar compact manifold S^2 with its standard conformal class, it was expected to be a simple case of the more involved problem: "On a complete n-dimensional manifold, describe the scalar curvature functions in a given conformal class." (For recent developments on the subject, see [**2, 4** and **5**], and the announcement [**15**] by R. Schoen of the solution of the Yamabe conjecture.)

In [**13**], J. Moser showed that any antipodally symmetric function on S^2 which is positive somewhere is the curvature of a metric in the standard conformal class. This result relies on a sharp form of a refined Sobolev estimate due to N. Trudinger (cf. [**17**]). A little later (cf. [**8**]), J. L. Kazdan and F. Warner showed that, even if they were positive somewhere (as made necessary by the Gauss-Bonnet theorem), monotonic functions of the distance to a point were forbidden as curvature functions. This followed from an identity satisfied by first spherical harmonics on S^2, which, when correctly stated, generalized to the n-sphere S^n. More recently, T. Aubin pushed further sharp Sobolev estimates for constrained functions by considering certain submanifolds of the Sobolev space $H^1(M)$. In this way, in [**1**], he

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obtains some new existence results of a rather intricate nature, solving Nirenberg's problem "up to" a first spherical harmonic.

The main results contained in this article are the following. The identity due to J. L. Kazdan and F. Warner alluded to above is shown to be rather universal. (This is our Theorem II.9.) *On any compact n-manifold M*, an identity ties together curvature functions and conformal vector fields as follows: *for any conformal vector field X of a Riemannian metric g, the scalar curvature s_g satisfies*

$$\int_M (X \cdot s_g) v_g = 0.$$

(*Here, v_g is the volume element of the metric g.*) Its proof relies on defining properly an action of the group of conformal transformations on the space of functions, then showing that the scalar curvature mapping nicely intertwines this action and the standard action. (This is done in §II.) It generalizes the Kazdan-Warner relation in two ways, firstly to a *general Riemannian* manifold, and secondly to *the whole Lie algebra of conformal vector fields.* (The gradients of first spherical harmonics generate only one part of the conformal Lie algebra of the standard sphere, the missing part corresponding to isometric vector fields.)

This last point turns out to be crucial in the next result we present in this paper, namely *the existence of new forbidden functions on the standard sphere.* These functions are not obstructed by the Kazdan-Warner conditions. (See §III, and the Appendix for details on the construction.) They disprove the conjecture made on p. 187 of [**7**]. This suggests that Nirenberg's problem is more connected with conformal transformations than merely with first spherical harmonics. The standard conformal class on the sphere appears then as special because it is the only one with a noncompact automorphism group. (This was proved independently by J. Ferrand, cf. [**11**], and by M. Obata, cf. [**14**], and is known as the solution of the Lichnerowicz conjecture.)

To underline the role of conformal transformations, we take up T. Aubin's refinement of Sobolev inequalitites for constrained functions. In particular, we give some new existence results, such as the occurrence of certain third order (or more generally any order larger than one) spherical harmonics as curvature functions in the standard conformal class of the sphere S^2. This is done in §IV. (Notice that the case of even order spherical harmonics is covered by J. Moser's result.)

To summarize, our contribution to Nirenberg's problem is to show that it is indeed much more complicated than expected, the forbidden functions being very likely quite numerous. On the other hand, the action of the conformal group which is one of our main tools in this article can be of help in discussing it.

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I. A quick review of conformal transformations.

I.1 Let M be a smooth compact n-dimensional manifold without boundary. We say that two metrics $\tilde{g}$ and g on M are *conformally equivalent* if there exist a diffeomorphism φ and a positive smooth function ρ such that

$$\tilde{g} = \rho^2 \varphi^* g. \tag{I.2}$$

If in (I.2), φ can be taken to be the identity, we say that $\tilde{g}$ and g are *conformally related.* We denote by $G = [g]$ the *conformal class* of g, i.e., the set of metrics conformally related to g.

I.3 A *conformal transformation* of g is a diffeomorphism φ of M such that $\varphi^\star g$ and g are conformally related. We *denote* the set of those transformations by $C(M,G)$ since it is obvious that it is an invariant of the conformal class G of g. It is a classical fact (cf. [**10**, p. 310]) that $C(M,G)$ is a finite dimensional Lie group. The only compact manifold on which one finds a conformal class with a noncompact group of conformal transformations is the sphere S^n for any of the conformal classes defined by metrics of constant sectional curvature.

I.4 We take as definition of a *conformal vector field* on a Riemannian manifold (M,g) that it is a vector field X whose flow $(\xi_t)_{t\in\mathbf{R}}$ is made of conformal transformations. If the diffeomorphisms ξ_t are isometries of the metric g, the vector field X is said to be a *Killing field* (or an infinitesimal isometry).

Conformal vector fields form a Lie algebra $\mathfrak{C}(M,G)$. It is a Lie subalgebra of the Lie algebra $\mathcal{T}M$ of vector fields of M. For M compact, $\mathfrak{C}(M,G)$ is the Lie algebra of $C(M,G)$.

One easily sees that the differential operator γ_g defined on $\mathcal{T}_M$ whose kernel is $\mathfrak{C}(M,G)$ takes its values in g-traceless symmetric 2-tensor fields, and is defined as follows

$$\gamma_g(X) = \mathcal{L}_X g - (2/n)(\operatorname{div}_g X)g. \tag{I.5}$$

(Here, $\mathcal{L}_X$ *denotes* the Lie derivative operator with respect to X and div_g the divergence operator given by $(\operatorname{div}_g X)v_g = \mathcal{L}_X v_g$ for the Riemannian volume element v_g.)

Notice that the notion of a g-traceless symmetric 2-tensor field only depends on the conformal class G.

I.6 The operator γ_g has a nice behavior in the conformal class G, namely, for $\tilde{g} = \rho^2 g$,

$$\gamma_{\tilde{g}} = \rho^2 \gamma_g. \tag{I.7}$$

(The proof of (I.7) is straightforward, and left to the reader. Notice also that $\operatorname{div}_{\tilde{g}} = \operatorname{div}_g + n\, d\log\rho$.)

From this property, it is clear that the kernel of γ_g remains indeed constant when g varies in the conformal class.

I.8 The conformal Lie algebra of the standard conformal structure on S^n that we denote by $\mathfrak{C}(S^n,C)$ splits as the direct sum

$$\mathfrak{C}(S^n,C) = \mathfrak{O}_{n+1} \oplus \mathfrak{PC}_{n+1}$$

where $\mathfrak{O}_{n+1}$ is the Lie algebra of the orthogonal group O_{n+1} corresponding to Killing fields of the metric c and $\mathfrak{PC}_{n+1}$ is an $(n+1)$-dimensional vector space which can be identified with the space of gradient vector fields of first spherical harmonics of (S^n,c). The decomposition above defines a $\mathbf{Z}_2$-grading of $\mathfrak{C}(S^n,C)$.

In $\mathfrak{C}(S^n,C)$, elements of $\mathfrak{PC}_{n+1}$ are orthogonal to Killing fields with respect to the global scalar product. This is why we call them *purely conformal* vector fields.

Using geodesic coordinates (θ,r) on (S^n,c) (r *denotes* the geodesic distance to a point chosen as north pole and θ a point on the equator), one easily shows that the flow $(\xi_t)_{t\in\mathbf{R}}$ of a purely conformal vector field is given by

$$\xi_t(\theta_0, r_0) = (\theta_0, 4\pi^{-1} r_0 \operatorname{Arctan} e^{-t}).$$

I.9 We derive a skew action of the conformal group on the space $\mathcal{F}M$ of functions on M by identifying $\mathcal{F}M$ with the conformal class G. Of course, this means that we made a choice of a background metric g in the conformal class. First, for φ in $C(M,G)$ we set

$$\text{for } n = 2, \qquad \varphi^*(g) = e^{2\alpha_2(\varphi)} g,$$
$$\text{for } n \geq 3, \qquad \varphi^*(g) = (\alpha_n(\varphi))^{4/(n-2)} g.$$

We now have the formulas:

$$\text{for } n = 2, \qquad (\varphi, e^{2u} g) \mapsto \varphi^*(e^{2u} g) = e^{2(u\circ\varphi + \alpha_2(\varphi))} g;$$
$$\text{for } n \geq 3, \qquad (\varphi, u^{4/(n-2)}) \mapsto \varphi^*(u^{4/(n-2)} g) = ((u\circ\varphi)\alpha_n(\varphi))^{4/(n-2)} g.$$

It will be convenient to *denote* this action by $\square_n$, so that $u \,\square_2\, \varphi = u\circ\varphi + \alpha_2(\varphi)$ and $u \,\square_n\, \varphi = (u\circ\varphi)\alpha_n(\varphi)$ if $n \geq 3$.

The linearized action is now given by the following lemma, the proof of which is straightforward.

I.10 LEMMA. *If $(\xi_t)_{t\in\mathbf{R}}$ is the flow of a conformal vector field X, then, for any smooth function u ($u > 0$ if $n \geq 3$),*

$$\frac{d}{dt}(u \,\square_2\, \xi_t)|_{t=0} = X \cdot u + \frac{1}{2}\operatorname{div}_g X,$$
$$\frac{d}{dt}(u \,\square_n\, \xi_t)|_{t=0} = X \cdot u + \frac{n-2}{2n}(\operatorname{div}_g X)u \quad \text{if } n \geq 3.$$

II. A conservation law in a conformal class involving the scalar curvature.

II.1 Let $\tilde{g}$ and g be two conformally related metrics. For questions involving the scalar curvature that we are going to consider, it is convenient (and classical) to write the conformal factor as we did in I.9 in the following way:

$$\text{(II.2)} \qquad \begin{cases} \rho = e^u & \text{for } n = 2, \\ \rho = u^{2/(n-2)} \text{ with } u > 0 & \text{for } n \geq 3. \end{cases}$$

If s_g *denotes* the scalar curvature of the metric g, one then has the classical formulas (see [**14**] for details)

$$\text{(II.3)} \qquad \begin{cases} s_{\tilde{g}} = (2\Delta_g u + s_g)e^{-2u} & \text{for } n = 2, \\ s_{\tilde{g}} = \left(4\dfrac{n-1}{n-2}\Delta_g u + s_g u\right) u^{-(n+2)/(n-2)} & \text{for } n \geq 3 \end{cases}$$

(where Δ_g denotes the Laplace-Beltrami operator attached to the metric g).

II.4 We are thus led to introduce the family of quasi-linear differential operators F_n on the space $\mathcal{F}M$ of smooth functions defined as

$$F_2(u) = e^{-2u}(2\Delta_g u + s_g),$$
$$F_n(u) = u^{-(n+2)/(n-2)}\left(4\frac{n-1}{n-2}\Delta_g u + s_g u\right) \quad \text{for } n \geq 3.$$

For any function U viewed as sitting in the tangent space $T_u\mathcal{F}M$, the *directional derivative* $U \cdot F_n$ is given by

$$(U \cdot F_2)(u) = e^{-2u}(\Delta_g U - (2\Delta_g u + s_g)U), \tag{II.5}$$

$$(U \cdot F_n)(u) = u^{-\frac{n+2}{n-2}} \left(\Delta_g U - \left(\frac{n+2}{n-2}u^{-1}\Delta_g u + \frac{1}{n-1}s_g\right) U\right) \qquad (n \geq 3). \tag{II.6}$$

II.7 LEMMA. *The divergence of a conformal vector field X satisfies the identity*

$$\Delta_g(\operatorname{div}_g X) = \frac{1}{n-1}s_g \operatorname{div}_g X + \frac{n}{2(n-1)}X \cdot s_g. \tag{II.7}$$

II.8 Formula (II.7) can be found in [**12**, p. 134]. Notice that an alternative proof follows by considering the linearizations of the maps F_n at $u \equiv 0$ for $n = 2$, and at $u \equiv 1$ for $n \geq 3$, since

$$\frac{d}{dt}(s_{\xi_t^*(g)})|_{t=0} = \frac{d}{dt}(s_g \circ \xi_t)|_{t=0} = X \cdot s_g$$

where $(\xi_t)_{t\in\mathbf{R}}$ denotes the flow of the conformal vector field X.

II.9 THEOREM. *For any conformal vector field X on a compact Riemannian manifold (M, g), the following identity called (N_X) holds*

$$\int_M X \cdot s_g\, v_g = 0. \tag{II.9}$$

(*Here, s_g denotes the scalar curvature of the metric g.*)

PROOF. The statement is about conformal classes of metrics on the compact manifold M. It is convenient to treat separately the cases $n = 2$ and $n \geq 3$.

II.10 The case $n \geq 3$ follows directly by integrating (II.7) against v_g since the left-hand side gives 0 and the right-hand side $\int_M X \cdot s_g v_g$ with the coefficient $(n-2)/2(n-1)$ which is not zero in the case at hand.

II.11 The case $n = 2$ is more delicate. A rough idea of the proof is to use the fact that the map F_2 intertwines the ordinary action and the action $\square_2$ of the conformal group on the space of functions. Notice that this fact was already behind the proof of Lemma II.7 which solved the case $n \geq 3$.

We take a metric g_0 as origin of a conformal class, and we use a subscript 0 for all objects attached to it.

If $g = e^{2u}g_0$ is a metric conformally related to g_0, then $s_g = F_2(u)$.

If φ is a conformal transformation, then

$$F_2(u \,\square_2\, \varphi) = s_{\varphi^*(e^{2u}g_0)} = F_2(u) \circ \varphi.$$

We evaluate the derivative for the flow $(\xi_t)_{t\in\mathbf{R}}$ of a conformal vector field at $t = 0$

$$\frac{d}{dt}(F_2(u \,\square_2\, \xi_t))|_{t=0} = \left(\frac{d}{dt}(u \,\square_2\, \xi_t)|_{t=0} \cdot F_2\right)(u) = X \cdot s_g,$$

hence,

$$X \cdot s_g = 2\left(\Delta_0\left(X \cdot u + \frac{1}{2}\operatorname{div}_0 X\right) - \left(X \cdot u + \frac{1}{2}\operatorname{div}_0 X\right)(2\Delta_0 u + s_0)\right)e^{-2u}.$$

Since M is compact, one can integrate this identity against v_g, the volume element of g. (Recall that $v_g = e^{2u}v_0$.) We get

$$\begin{aligned}\int_M X\cdot s_g v_g = & -4\int_M X\cdot u\,\Delta_0 u\, v_0 - 2\int_M s_0 X\cdot u\, v_0\\ & -2\int_M \mathrm{div}_0\, X\,\Delta_0 u\, v_0 - \int_M s_0\,\mathrm{div}_0\, X\, v_0,\end{aligned}$$

and after integrating by parts

$$\begin{aligned}\int_M X\cdot s_g v_g = & -4\int_M X\cdot u\,\Delta_0 u\, v_0 + 2\int_M \mathrm{div}_0\, X\, s_0 u\, v_0 + \int_M u\, X\cdot s_0\, v_0\\ & -2\int_M \Delta_0 \mathrm{div}_0\, X\, u\, v_0 + \int_M X\cdot s_0\, v_0.\end{aligned}$$

Now, taking account of identity (II.7) leaves us with

$$\int_M X\cdot s_g\, v_g - \int_M X\cdot s_0\, v_0 = -4\int_M X\cdot u\,\Delta_0 u\, v_0.$$

We shall show in Lemma II.12 (a slight extension of (8.3) in [**8**] that we present below) that the integral on the right-hand side vanishes. Therefore, the integral $\int_M X\cdot s_g v_g$ does not depend on a metric in the conformal class. Two cases are then to be consider. Either the connected component of the identity of the conformal group $C_0(M,G)$ is compact, hence one can find a metric g_1 within the conformal class admitting it as a group of isometries. It is then clear that $X\cdot s_{g_1} = 0$, and the integral vanishes. Or $C_0(M,G)$ is noncompact, and by the theorem of Obata-Ferrand (M,G) is the standard conformal sphere (S^2,C). The integral vanishes also in this case, since $s_c \equiv 2$. □

II.12 LEMMA. *For any conformal vector field X and any smooth function u on a compact surface M, one has*

$$\int_M \Delta_g u\,(X\cdot u)\, v_g = 0. \tag{II.12}$$

PROOF. It is based on the conformal invariance of the Dirichlet integral $\int_M g^{-1}(du,du)v_g$. (We deliberately wrote g^{-1} the metric on 1-forms to emphasize that for a metric $e^{2u}g$ the square norm of a 1-form is multiplied by e^{2u}.) We now let a conformal flow act

$$\begin{aligned}\int_M g^{-1}(du,du)v_g &= \int_M e^{-2\alpha_2(\xi_t)}g^{-1}(du,du)e^{2\alpha_2(\xi_t)}v_g\\ &= \int_M (\xi_t^* g)^{-1}(du,du)v_{\xi_t^* g}\\ &= \int_M g^{-1}\left(d(u\circ\xi_{-t}), d(u\circ\xi_{-t})\right) v_g.\end{aligned}$$

Taking the derivative of this identity at $t=0$ gives Formula (II.12). □

II.13 REMARK. Another proof of Theorem II.9 can be found in [**4**]. This other proof explains why the cases $n=2$ and $n\geq 3$ behave so differently. Also, it shows that formula (II.9) is one among an infinite family of conservation laws attached to the conformal group.

II.14 Theorem II.9 is the proper generalization of the Kazdan-Warner integrability condition (cf. [**8**]) which was valid for the pure conformal vector fields of the standard conformal sphere. We see that it is by no means a special identity due to peculiarities of the sphere, but truly a universal relation.

III. Forbidden functions attached to a conformal class. New examples.

III.1 We fix a conformal class G on a compact manifold M. We say that a smooth function u is *forbidden* if u cannot be the scalar curvature of a metric belonging to G.

It follows trivially from Theorem II.9 that a function u for which there exists a conformal vector field X so that for any metric g in G $\int_M X \cdot u\, v_g \neq 0$ is forbidden. Since v_g can be any positive density, this will be a priori so only if $X \cdot u$ keeps a fixed sign on M. Notice though that this can never be the case when $C_0(M, G)$ is compact, since then all conformal vector fields X can be made Killing fields for the same metric g_0 in G, and for the volume element v_0 of g_0

$$\int_M X \cdot u\, v_0 = -\int_M u \operatorname{div}_0 X\, v_0 = 0$$

since $\operatorname{div}_0 X = 0$. Therefore, (II.9) provides forbidden functions only on (S^n, C), although it is an integrability condition for all (M, G).

III.2 By its very definition, the set of forbidden functions attached to a conformal class G is invariant under the conformal group $C(M, G)$.

Notice also that if u has been shown to be forbidden by condition (II.9) in the obvious way that we presented above, the same is true for all functions $\chi \circ u$ where χ is a monotonic function on the real line.

III.3 The main point we would like to make in this section is that in condition (II.9) conformal vector fields other than purely conformal ones have also to be taken into account. We illustrate this point by giving a counterexample to a conjecture of J. L. Kazdan (cf. [**7**, p. 187]). In [**8**], the forbidden functions $h + h_0$ where h is a first spherical harmonic on S^n and h_0 a constant were exhibited thanks to the condition $(N_{\nabla^c h})$ (where $\nabla^g h$ *denotes* the gradient of h for the metric g). More generally, the condition $(N_{\nabla^c h})$ prohibits all monotonic functions of the distance to the pole associated to h.

To get new forbidden functions for the standard conformal class C on S^2, we consider the vector field $X_\alpha = \cos\alpha\, \nabla^c h + \sin\alpha\, Y$ where Y is the Killing field deduced from $\nabla^c h$ by a $\pi/2$-rotation which is obviously conformal. With our choice, $\nabla^c h$ and Y both vanish at the north and south poles. If we construct a function u so that $X_\alpha \cdot u \geq 0$ everywhere $(X_\alpha \cdot u \not\equiv 0)$, we get a new forbidden function because of the following lemma.

III.4 LEMMA. *The conformal vector field X_α is the gradient of a first spherical harmonic with respect to a metric belonging to C only if $\alpha = 0 \pmod{\pi}$.*

PROOF. Suppose that $X_\alpha = \nabla^g k$ where ∇^g is the gradient operator for a metric $g = e^{2u}c$. We then have

$$\cos\alpha\, \nabla^c h + \sin\alpha\, Y = e^{-2u} \nabla^c k,$$

which in a covariant form reads

$$e^{2u}(\cos\alpha\, dh + \sin\alpha\, Y^\flat) = dk.$$

(Here, we used $\flat$ to denote the index lowering operator for c.) Therefore, by taking the exterior differential,

$$2\,du \wedge (\cos\alpha\, dh + \sin\alpha\, Y^\flat) = -\sin\alpha\, dY^\flat.$$

In particular, at the north and south poles where dh and Y vanish, we have $\sin\alpha\, dY^\flat = 0$. If $\sin\alpha \neq 0$, this says that at the north and south poles $DY = 0$ where D denotes the Levi-Civita covariant derivative. (Indeed, the symmetric part of DY vanishes since Y is a Killing field, and the previous equation precisely says that the skew-symmetric part of DY vanishes at those points.)

But, along any geodesic the Killing field Y satisfies the Jacobi equation, hence vanishes on the whole sphere, a contradiction.

III.5 We now claim that one can construct a function f with support in a small ball away from the poles so that $h + f$ is forbidden by the conformal vector field X_α for some angle α, $0 < \alpha < \pi/2$, because

$$X_\alpha \cdot (h + f) = \sin\alpha\, Y \cdot f + \cos\alpha\, c(\nabla^c h, \nabla^c f) + \cos\alpha\, c(\nabla^c h, \nabla^c h)$$

keeps a fixed sign on S^2. The rough idea is to take f to be the height function for a cone with basis the small ball mentioned above and with vertex appropriately placed. Since the construction is a bit technical, we detail it in an appendix.

Therefore, we proved the following theorem.

III.6 THEOREM. *There are forbidden functions attached to the standard conformal class of S^2 which are not obviously forbidden by the condition (N_X) where X varies amongst gradients of first spherical harmonics.*

III.7 REMARKS. (i) It may be that, by considering the relative measure of the sets of S^2 where the functions $c(\nabla^c h, \nabla^c(h+f))$ and $Y \cdot f$ are respectively positive and negative, one might deduce that condition $(N_{\nabla^c h})$ is necessarily violated. This seems to be difficult to put to work.

(ii) Our construction, suitably modified, generalizes to higher dimensional spheres. The only point to clarify is what happens to Killing fields when going from even-dimensional spheres to odd-dimensional spheres. One must allow the Killing field Y to have two curves of zeroes. We do not detail this here.

IV. Some remarks on the images of the maps F_n.

IV.1 We pointed out in II.11 that the map F_2 intertwines the actions $\square_2$ and $\circ$ on functions. It is straightforward to show that for $n \geq 3$ one has, for any positive function u and any conformal transformation φ,

$$F_n(u \,\square_n\, \varphi) = F_n(u) \circ \varphi.$$

In particular, this shows that the orbit of the constant functions $u \equiv 0$ for $n = 2$ and $u \equiv 1$ for $n \geq 3$ is $\{\alpha_n(\varphi), \varphi \in C(M, G)\}$. This orbit is mapped by F_n onto the ordinary orbit under $C(M, G)$ of s, the scalar curvature of the metric g taken as origin of the conformal class. If g has constant scalar curvature, this orbit is reduced to the point $\{s\}$. A very special case of this is provided by taking g to be a metric with constant sectional curvature 1 in the standard conformal class C of S^n.

The counterimage of $s_c = n(n-1)$ by F_n is reduced to the $\square_n$-orbit of the constant function 1 if $n \geq 3$ or 0 if $n = 2$ as follows from a theorem of M. Obata (cf. [**14**]), namely

IV.2 THEOREM. *The conformal class of an Einstein metric on a compact Riemannian manifold (M, g) contains no other metric with constant scalar curvature except if (M, g) is isometric to a standard sphere (S^n, c).*

Here, we include a proof of it which is more transparent and follows the lines of thought of this article.

PROOF. If z_g denotes the traceless part of the Ricci curvature of a Riemannian metric g, then

$$\gamma_g^* z_g = (2/n - 1)\, ds_g \tag{IV.3}$$

where, for a symmetric 2-tensor field h, one remarks that

$$\gamma_g^* h(x) = -2 \sum_{i=I}^{n} (D_{e_i} h)(e_i, x)$$

for any tangent vector x and orthonormal basis (e_i).

(The classical version of the second Bianchi identity for the full Ricci curvature r_g is $\gamma_g^* r_g = ((2-n)/n)$. It is a trivial exercise to deduce (IV.3) from it since $z_g = r_g - n^{-1} s_g g$.)

We now notice that for two conformally related metrics g_0 and $g = \rho^{-2} g_0$, one has (cf. [**14**, p. 255])

$$z_g = z_{g_0} + (n-2)\rho^{-1}\gamma_{g_0}(\nabla^{g_0}\rho).$$

If the metric g_0 is Einstein, the expression simplifies, and one gets

$$(n-2)^{-1} z_g = \rho^{-1}\gamma_{g_0}(\nabla^{g_0}\rho).$$

Therefore,

$$(n-2)^{-1}\gamma_g^* z_g = \gamma_g^*\left[\rho^{-1}(\gamma_{g_0}(\nabla^{g_0}\rho))\right].$$

Since we assumed that g has constant scalar curvature, thanks to (IV.3), we get

$$\gamma_g^*[\rho^{-1}(\gamma_{g_0}(\nabla^{g_0}\rho))] = 0.$$

We now take the L^2-inner product for the metric g of this identity with $\nabla^{g_0}\rho$. We obtain

$$\int_{S^n} g(\gamma_g^*(\rho^{-1}\gamma_{g_0}(\nabla^{g_0}\rho)), \nabla^{g_0}\rho) v_g = 0.$$

Using relation (I.7) to the effect that $\gamma_g = \rho^{-2}\gamma_{g_0}$ and the definition of the adjoint, we end up with

$$\int_{S^n} \rho^{-1} g(\gamma_g(\nabla^{g_0}\rho), \gamma_g(\nabla^{g_0}\rho)) v_g = 0.$$

Since $\rho > 0$, this ensures that $\gamma_g(\nabla^{g_0}\rho) = 0$. Therefore, $z_g \equiv 0$, and g is an Einstein metric. Notice that this finishes the proof of Obata's theorem since (S^n, c) is conformally flat.

In the more general case, we notice that, according to (II.7) the function ρ is an eigenvalue of the Laplacian of the metric g for the eigenvalue $s_{g_0}/(n-1)$, hence, by the Lichnerowicz-Obata theorem, the Einstein metric g_0 is isometric to c on S^n. □

IV.4 Besides the case of constant functions whose orbits by the conformal group are degenerate to a point, hence cannot be of much help to describe the image of the maps F_n, the orbits under the action by $C(S^2, C)$ allow us to prove the following result.

IV.5 PROPOSITION. *Any smooth function on S^2 which is positive somewhere belongs to the L_p-closure of the image of F_2 for all $p \geq 1$.*

PROOF. Suppose u is a smooth function positive at m that we take to be the north pole on S^2. For any $\varepsilon > 0$, there exists a conformal transformation φ of (S^2, C) which maps the cap $B_{\eta+\pi/2}$ of radius $\eta + \pi/2$ containing the northern hemisphere of S^2 into a ball centered at m whose volume is less that ε. (Compare for example with the formula in I.8.)

We now set

$$\bar{u} = u \quad \text{on } S^2 - \varphi(B_{\eta+\pi/2});$$
$$\bar{u} = u \circ \varphi \circ \tau \circ \varphi^{-1} \quad \text{on } \varphi(B_{-\eta+\pi/2})$$

where τ is the antipodal map on S^2; along the geodesic through m, $\bar{u}$ interpolates linearly for the $\varphi^* c$-distance between its values on the boundaries of the balls $\varphi(B_{\eta+\pi/2})$ and $\varphi(B_{-\eta_\pi/2})$ for the rest of S^2.

By construction, the function $\bar{u}$ is continuous, invariant under the antipodal map $\varphi \circ \tau \circ \varphi^{-1}$ of the metric $\varphi^* c$.

Moreover,

$$\|u - \bar{u}\|_{L^p}^p = \int_{B_{\eta+\pi/2}} |u - \bar{u}|^p \leq 2^p \|u\|_{C^0}^p \varepsilon.$$

One can then find a smooth function $\tilde{u}$ invariant under the antipody of the $\varphi^* c$-metric ε-close to $\bar{u}$ in the C^0-topology. Therefore, $\tilde{u}$ is positive somewhere. By J. Moser's theorem, $\tilde{u}$ is the curvature of some metric on S^2 conformal to $\varphi^* c$, hence to c. Clearly, the distance in the L^p-norm between u and $\tilde{u}$ can be made arbitrarily small. □

IV.6 In [**1**], T. Aubin shows that any function on S^2 which is positive somewhere belongs to the image of F_2 provided one adds to it some first order spherical harmonic i.e., an eigenfunction for the lowest eigenvalue of a standard metric. We show in the next theorem that the first spherical harmonics do not play any specific role with respect to this problem.

IV.7 PROPOSITION. *Let f be a smooth function on S^2 which is positive somewhere. Then, for any $p \geq 1$ there exists at least one spherical harmonic of order p, say $h^{(p)}(f)$, so that $f + h^{(p)}(f)$ belongs to the image of F_2.*

PROOF. We do not duplicate the arguments in [**1**]. To prove that the equation $f + h^{(p)}(f) = F_2(u)$ admits a solution, we only need to show that for each p $(p \geq 1)$, one can find a set $(h_i^{(p)})$ of spherical harmonics of order p which form an admissible set. (This means that the h_i's change sign, and that $\sum_i |h_i^{(p)}| > 0$ at each point of the sphere.) This is done in Lemma IV.8. In such a case, the best constant in Trudinger's extension of the Sobolev inequality can be cut in half, and the direct variational method works on a submanifold of the Sobolev space H^1 defined by constraints involving the admissible set.

IV.8 LEMMA. *For any $p \geq 1$, the space V_p of spherical harmonics of order p on the sphere S^2 contains a subspace of dimension 3 whose elements do not all vanish at the same point.*

PROOF. As is well known, the spaces V_p are irreducible representation spaces of O_3, the isometry group of (S^2, c). In each V_p, there are some special eigenfunctions which depend only on the distance to some point. (E. Cartan calls them the *zonal* eigenfunctions.) We call this point the *pole* of the zonal function considered since its antipodal point plays the same role. These functions form an orbit of O_3, parametrized by the action on the poles. For each p, the set of zeroes of a zonal spherical harmonic of order p is exactly a union of p circles of radii $(r_i)_{1\leq i\leq p}$ all centered at the pole. By fixing arbitrarily two such poles, one is left with at most finitely many common zeros, say $(m_j)_{1\leq j\leq N}$, for the two zonal spherical harmonics $h_1^{(p)}$ and $h_2^{(p)}$. If any other zonal spherical harmonic of order p has a common zero with $h_1^{(p)}$ and $h_2^{(p)}$, this would mean that the function $m \mapsto \prod_{i,j}(d(m,m_j) - r_i)$ is identically zero, a contradiction. (Here, d *denotes* the distance between two points.) □

IV.9 COROLLARY. *For any $p \geq 2$, there are spherical harmonics of order p which are in the image of F_2.*

IV.10 REMARKS. (i) Of course, as we said in 0.3, all spherical harmonics of even order are in the image of F_2 by J. Moser's theorem.

(ii) What makes the first spherical harmonics special is the fact that one needs a basis of them to define an admissible set, hence leaving no place to solve this equation with f a first spherical harmonic.

(iii) It is interesting to remark that the set of forbidden functions has no linear structure. Namely, if f is a first spherical harmonic and φ a conformal transformation, then by T. Aubin's theorem there is some first spherical harmonic $h^{(1)}(f\circ\varphi)$ so that $f\circ\varphi + h^{(1)}(f\circ\varphi)$ lies in the image of F_2 although both $f\circ\varphi$ and $h^{(1)}(f\circ\varphi)$ are forbidden functions.

A. Appendix.

A.0 The function f quoted in III.5 is a smoothing of the height function f_0 of a cone with circular basis and with an appropriately placed vertex.

A.1 We work in a Euclidean 2-dimensional space $(\mathbf{R}^2, e)$ tangent to S^2 at some point and we approximate $\nabla^c h$ (where h is a first spherical harmonic on S^2) by a constant vertical vector field, i.e., the Euclidean gradient of the linear function ky in that plane for some real number k. This is because we think of our construction as being done in a small ball. In the (x, y)-plane, we call B the projection of the vertex A of the cone. The point B has polar coordinates (b, β), a point μ on the boundary circle (r, θ).

A.2 The gradient of f_0 lies tangentially to the cone, hence at each point P it is collinear to the segment joining P to the vertex A and points towards A. Moreover, its norm which is constant along this segment measures its slope. If a *denotes* the height of the cone, one has $\nabla^e f_0 = a\overrightarrow{\mu A}/|\mu A|^2$ so that its components in the (x, y)-plane are

$$\left(a\frac{r\cos\theta - b\cos\beta}{r^2+b^2-2br\cos(\theta-\beta)}, a\frac{r\sin\theta - b\sin\beta}{r^2+b^2-2br\cos(\theta-\beta)}\right).$$

We want to control the sign of the function $z = k\,e(\nabla^e y, \nabla^e(ky + f_0))$ in the following sense. We want this function to change sign (as we want the function $c(\nabla^c h, \nabla^c(h+f))$ not to be obviously forbidden by condition $N_{(\nabla^c h)}$), but we still want to be able to find a vector field $X_\alpha^0 = k(\cos\alpha\,(\partial/\partial y) - \sin\alpha\,(\partial/\partial x))$ so that $X^0 \cdot (ky + f_0)$ keeps a fixed sign.

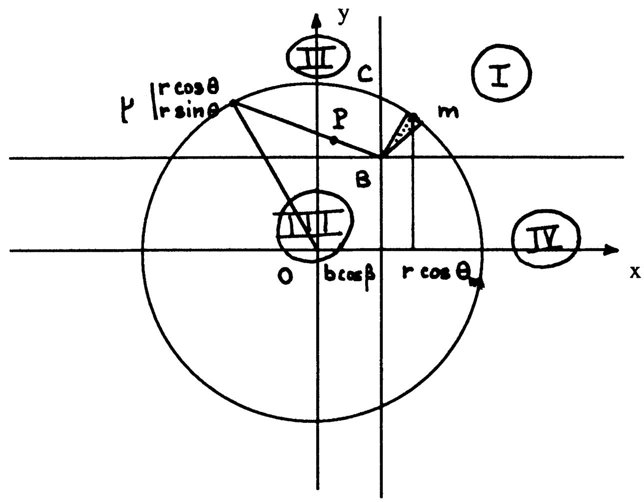

FIGURE 1.1

A.3 Therefore, we must study how the function

$$z(\theta) = ak\frac{r\sin\theta - b\sin\beta}{r^2 + b^2 - 2br\cos(\theta-\beta)} + k^2$$

depends upon the parameters a, b, k and β. For that purpose, it is useful to look at Figure 1.

We want the function z to achieve its minimum in a small region $\mathcal{A}^-$ located to the right of the y-axis going through B, the constant k being taken large enough so that z is positive elsewhere. We are thus led to compare the coordinates of the point ρ_m on the circle where z achieves its minimum to those of the point C on the base circle with y-coordinate $b\cos\beta$. We must have the polar coordinates (r, θ_m) of μ_m satisfy $r\cos\theta_m > b\cos\beta$.

A.4 We know that (r, θ_m) are solutions of the equation

$$(r^2 + b^2\cos 2\beta)\cos\theta_m + b^2\sin 2\beta\sin\theta_m - 2br\cos\beta = 0.$$

This relation can also be written

$$r\cos\theta_m - b\cos\beta = b\left(\cos\beta - \frac{b}{r}\cos(2\beta - \theta_m)\right).$$

The condition we mentioned before ($r\cos\theta_m > b\cos\beta$) will therefore be satisfied if $\cos\beta > b/r$, a possible explicit value being $\beta = \pi/4$, $b/r = \frac{1}{2}$.

In such a case, by choosing appropriately k and a, z will be negative in a small zone $\mathcal{A}^-$ around the minimum whose size can be made arbitrarily small. Therefore, the vector field $\nabla^e(ky + f_0)$ makes an angle less than $\pi/2$ with $\partial/\partial y$ except on $\mathcal{A}^-$.

A.5 To prove existence of the vector field X_α^0, we analyse the situation in the four quadrants centered at B as shown in Figure 1. For each zone, we consider the direction in which the vector $\nabla^e(ky + f_0)$ points. To distinguish between the affine quadrants (where points are located) and the vector quadrants (in which vectors point), we use respectively roman and arabic figures.

In zone II, $\nabla^e(ky + f_0)$ belongs by construction to quadrant 1. Even more, since the y-component of $\nabla^e f_0$ is far away from its minimum, in zone II, the y-component of $\nabla^e(ky + f_0)$, i.e., the function z, is bounded from below by some positive number. Hence, $\nabla^e(ky + f_0)$ belongs to an angular sector containing the y-axis included in quadrant 1.

In zone III, $\nabla^e(ky + f_0)$ clearly belongs to the same angular sector of quadrant 1 mentioned above.

In zone IV, $\nabla^e(ky + f_0)$ obviously belongs to quadrant 2.

We now come to the more delicate description of what happens in zone I. In $\mathcal{A}^-$, z is by definition negative, but by choosing appropriately k can be made arbitrarily small. On the other hand, since $\mathcal{A}^-$ lies strictly in zone I, the x-component of $\nabla^e(ky + f_0)$ is bounded from above by a negative number. It then follows that $\nabla^e(ky + f_0)$ belongs to an angular sector as close as we like to the x-axis in quadrant 3. Outside $\mathcal{A}^-$ in zone I, z is positive, hence $\nabla^e(ky + f_0)$ belongs to quadrant 2.

A.6 We therefore proved that one can indeed find an angle α, $0 < \alpha < \pi/2$, so that $\nabla^e(ky + f_0)$ belongs to the upper-half plane making this angle with the x-axis. The function f_0 can be smoothed out while keeping its properties: at the basis of the cone by an ordinary radial mollification; at the tip of the cone, by using a vertex-centered mollification.

A.7 To go from this Euclidean description back to S^2 uses the uniform convergence of the Riemannian geometry (both the metric and the gradient operator) to the Euclidean geometry when the size of a ball goes to zero.

Notice though that the function that we produce is not small in absolute norm since the height of the cone a must be large enough to force the function z to have a negative minimum.

A.8 It is clear from the construction that one can superimpose any finite family of such functions f_0 with small disjoint supports. This clearly provides an infinite family of new forbidden functions.

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